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A description of the Shapley value using a binary procedure

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Abstract. The study of solution concepts in the theory of cooperative games aims to provide solutions to practical situations where the main problem is to specify a rule to divide a certain amount obtained with the cooperation of various players. The Shapley value is one of the fundamental solution concepts in the theory of cooperative games. In this article, we introduce a procedure to obtain the Shapley value using the basic principle of sharing a good between two parties. In particular, we provide a simple method to introduce the Shapley value and a recursive formula to compute it. We do this by providing a recursive procedure to extend the so-called standard solution to solve 2-player cooperative games. In our main result, we characterize the Shapley value using a single recursive formula which can be used to implement the Shapley value computationally.

Keywords: Solution concepts, standard solution, adjusted games, binary partitions.

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1 Introduction

One of the central topics in the theory of cooperative games is the study of solution concepts. Informally, if a cooperative situation between n players is modeled by a cooperative game, one is interested in finding a rule which assigns payment to every player in the game. In the axiomatic approach to find such a rule, one proposes various properties (axioms) and then determines the set of allocations satisfying these properties.

The Shapley value [1] is probably the most studied solution concept in all cooperative game theories. This solution has been characterized by various authors. Among the classic characterizations of the Shapley value, we find Chun's characterization [2] using efficiency, triviality, coalitional strategic equivalence, and fair ranking, Myerson's characterization [3] using the fair allocation rule and the balanced contribution axiom, and Young's characterization [4] using symmetry, efficiency, and strong monotonicity. Various recent developments are still related to the Shapley value. Among them, we can find the following. Aumann [5] showed that the Shapley value is related to solutions of classic problems in the antique passages of the Talmud [5]. Van den Brink showed [6] that changing null by nullifying players, in the original characterization of the Shapley value, leads to a characterization of the equal division solution (which distributes the worth of the grand coalition equally among all players in a cooperative game). Casajus [7] showed that the characterization of Young [4] continues to be valid by relaxing the axiom of symmetry with the axiom of sing symmetry. Casajus and Huettner [8] characterized the class of egalitarian Shapley values using efficiency, symmetry, and weak monotonicity.

Aside from the characterization, there is also interest in finding novel ways to compute the Shapley value. With this respect, in this note, we introduce a simple binary recursive method by means of which we can obtain the Shapley value in a direct way. This method allows us to establish a recursive formula to compute the Shapley value.

The starting point of our procedure is the standard solution for 2-player games: given a 2-player game, the standard solution of a player in the game is the sum of the amount obtained by the player alone and the equal surplus obtained by the cooperation of both players. Our procedure extends this solution to a solution for n-player games. Before we give more details about the extension, we provide an example of a situation where a generalization of the standard solution is needed.

Example 1.1. Two friends, A and B, were working on a science project in order to participate in a science contest. At a certain point in the development of the project, both friends needed the help of a third friend C to work out an idea on a field in which they are not experts. The three friends decided to cooperate, the project was finished, and they won a prize that is worth one unit. They want to divide the reward, and for this, they consider the portion of the project finished with their cooperation. A or B would do $1/4$ of the project working alone, C would do nothing working alone, A and B would do $3/4$ working together and, A and C, or B and C, would do $1/2$ working together. Before A and B needed the help of C they would use the standard solution to split the reward obtained. The problem is to split the reward obtained by the three friends using the standard solution. We can solve this problem using an extension of the standard solution to three-player games.

In fact, we provide a method to solve an n-player game using only the standard solution. In order to motivate our procedure, consider the following informal method to solve the problem of example 1.1. First, use the standard solution to split the total reward between $\{A, B\}$ and $\{C\}$. Second, use the standard solution to divide the amount of $\{A, B\}$ by $\{A\}$ and $\{B\}$. The same method can be applied if, in the first step, we use any of the other two possible binary partitions of $\{A, B, C\}$. Finally, since we have no preference over any of the three partitions, we study the solution given by the average of the three allocations obtained.

To formalize the previous method, we use the standard solution as follows. Given a n-player game and a nontrivial binary partition of the set of players we assign, to every coalition in the binary partition, the sum of the worth of the coalition and the equal surplus obtained by the cooperation of both coalitions. This binary rule can be applied recursively using adjusted games. Informally, in an adjusted game, the worth of the grand coalition is the amount assigned by the standard solution to this coalition. Thus, given a n-player game, we take a binary partition of the set of players and consider an adjusted game for every coalition in the binary partition. If we iterate this process for every adjusted game until only singletons remain, we end with a distribution of the worth of the grand coalition in the original game. Since the way in which one takes binary partitions is not unique, we obtain a set of allocations, one for each binary total partition [9] of the set of players, whose average provides a solution concept for n-player games. This solution turns out to be the Shapley value, and the described procedure allows us to establish a recursive formula that characterizes the Shapley value.

A similar binary procedure was introduced in [10] to propose the consensus value. Ju et al. considered a two-sided negotiation process where an entrant player and an existing coalition get what they do on their own and share equally the surplus obtained by the cooperation of both parties. This process is implemented recursively for every order in which the players enter the cooperation process. The consensus value is obtained averaging over all the resulting allocations. Thus, to derive the consensus value, one uses the permutations of the set of players while, in our procedure, we make use of the binary total partitions of the set of players[1].

The extension of solutions is a commonly used technique to propose new solutions concepts. For example, in [11], Bejan and Gómez suggest a family of extensions of the core which coincide with the core in case the game has a non-empty core. Regarding the extension or characterization of solutions related to the standard solution, Hart and Mas-Colell showed [12] that a solution φ is standard for 2-player games and consistent if and only if φ is the Shapley value. Ruiz, Valenciano and Zarzuelo proposed [13] the least square values (which contain the Shapley value) obtained minimizing a quadratic function. Ruiz et al. showed that every least-square value coming from a consistent weight function m is characterized by being standard for two-player games and m-consistent. More about the use of reduced games to characterize solutions can be found in [14], where Davis and Maschler introduced the kernel of a cooperative game and used a suitable reduced game to show how to find elements of the kernel for certain families of games. We also notice that, in [15], Ruiz et al. proposed a suitable reduced game to characterize the least square prenucleolus and the least square nucleolus, two solutions which are standard for two-player games, using the consistency axiom.

We observe that the extensions of the standard solution just reviewed require a specific reduced game which can be very technical or even artificial for some practical situations. We consider here an extension based on a method that is natural and easy to understand: it only requires the recursive application of the standard solution for binary partitions.

The rest of this paper is organized as follows. In Section 2, we recall some elementary game theory concepts, introduce the concept of the adjusted game, and recall the concept of binary total partition. In Section 3, we introduce an extension of the standard solution using adjusted games and binary total partitions. We show that the extension satisfies a recursive formula, and we prove that this formula is satisfied by a solution concept if and only if the solution concept is the Shapley value.

[1] These sets are not isomorphic, see the last paragraph of Section 2.2.

2 Preliminaries

A cooperative game in characteristic function form is a pair (N, v) where $N = \{1, 2, \dots, n\}$ is the set of players and $v: 2^N \to \mathbb{R}$ is a real function with $v(\emptyset) = 0$. The set 2^N is the power set of N, N is called the grand coalition, and $v(S)$ is the worth of the coalition $S \subset N$. Let G^n be the set of games (N, v) for which $|N| = n$ and let G be the union of the sets G^n over $n \in \mathbb{N}$. We refer to $(N, v) \in G$ as an n-player game. A solution concept φ is a function which assigns to every game $(N, v) \in G$, with $|N| = n$, a set of allocations $\varphi(N, v) \subseteq \mathbb{R}^n$. If for every $(N, v) \in G$ we have that $\varphi(N, v)$ is a point, then we say that φ is a value.

Notation. In what follows, we will use $S, T, U, \dots$ to denote coalitions and $s, t, u, \dots$ to denote the respective cardinalities. For brevity, we will use $v(1)$, $v(12)$, etc., instead of $v(\{1\})$, $v(\{1,2\})$, etc.

2.1 Adjusted games

Consider $(N, v) \epsilon G^n$ and a binary partition of N, $\{S, N \backslash S\}$. If we add the worth of a coalition, in this partition, and the equal surplus obtained by the cooperation of both coalitions, then S obtains $1/2(v(N) + v(S) - v(N \backslash S))$ and $N \backslash S$ obtains $1/2(v(N) + v(N \backslash S) - v(S))$. This is the standard solution of (N, v) and the given binary partition.

In order to share the previous amounts between the members of S and $N \backslash S$ using the standard solution, we introduce adjusted games. For $S \subset N$ define the adjusted game (S, v_a^N) by

$$v_a^N(T) = 1/2(v(N) + v(T) - v(N \backslash T)), \qquad T \subseteq S. \tag{1}$$

Notice that the worth of the grand coalition in the games (S, v_a^N) and $(N \backslash S, v_a^N)$, i.e., the amounts $v_a^N(S)$ and $v_a^N(N \backslash S)$ respectively[2], are the amounts which S and $N \backslash S$ obtain using the standard solution.

Once $v(N)$ has been distributed between S and $N \backslash S$ and the games (S, v_a^N) and $(N \backslash S, v_a^N)$ have been defined, the process of taking binary partitions and adjusted games can be iterated. To this end, we need to compose adjusted games. For $T \subset S \subset N$ we define the composition $(T, (v_a^N)_a^S)$, by

$$((v_a^N)_a^S)(U) = 1/2(v_a^N(S) + v_a^N(U) - v_a^N(S \backslash U)), \qquad U \subseteq T. \tag{2}$$

Hence if $\{T, S \backslash T\}$ is a binary partition of S then, composing adjusted games, T gets $(v_a^N)_a^S(T)$ and $S \backslash T$ gets $(v_a^N)_a^S(S \backslash T)$. We continue iterating this process for any non-singleton coalition until only singletons remain. At the end of the process, if $N \supset S_1 \supset \cdots \supset S_k \supset \{i\}$ then, using Equation (2) repeatedly, player i obtains the amount $\left(\cdots (v_a^N)_a^{S_1} \cdots \right)_a^{S_k}(i)$.

Example 2.1. Let $(N, v) \in G^3$. By using Equations (1) and (2), one obtains

$$\begin{aligned} ((v_a^N)_a^{\{1,2\}})(1) &= 1/2(v_a^N(12) + v_a^N(1) - v_a^N(2)) \\ &= 1/4\big(v(N) - v(23) + v(12) - v(2) + v(13) - v(3) + v(1)\big). \end{aligned}$$

Similar formulas can be obtained using other coalitions $S \subset \{1, 2, 3\}$.

2.2 Binary total partitions

Take a binary partition of $N = \{1, 2, \dots, n\}$. Next, take a binary partition of each non-singleton coalition. Continue taking binary partitions of each non-singleton coalition until only singletons remain. The finite sequence of partitions of N obtained with this process is called a binary total partition of N. The set of binary total partitions of N is in one-to-one correspondence with the set

[2] Observe that the game (S, v_a^N) considers coalitions of $T \subseteq S$ while the game $(N \backslash S, v_a^N)$ considers coalitions $T \subseteq N \backslash S$.

of unordered complete binary trees with n labeled endpoints. The number of such trees is the product $1 \cdot 3 \cdots (2n-3)$, for $n \geq 2$ (see [9], p. 14, example 5.2.6).

Let B_N be the set of binary total partitions of N and $p(n) = |B_N|$. Thus $p(1) = 1$ and

$$p(n) = 1 \cdot 3 \cdots (2n-3), \qquad n \geq 2\,. \tag{3}$$

Lemma 2.1. The sequence of Equation (3) satisfies the recursive formula

$$p(n) = \frac{1}{2}\sum_{i=1}^{n-1}\binom{n}{i}p(i)p(n-i)\,, \qquad n \geq 2\,. \tag{4}$$

Proof. Suppose $p(1), \dots, p(n-1)$ are known. There exists $\binom{n}{i}$ binary partitions of N, $\{S, N \setminus S\}$, such that $|S| = i$, $1 \leq i \leq n-1$. With each one of these partitions, we obtain $p(i)p(n-i)$ binary total partitions of N.

The sequence of Equation (3) satisfies $p(n) < n!$ for $2 \leq n \leq 5$ and $p(n) > n!$ for $n \geq 6$. In particular B_N is not isomorphic to the group of permutations of N for $n \geq 2$.

3 A binary extension of the standard solution

Let $(N, v) \in G^n$. Consider a binary partition of N, $\{S, N\setminus S\}$, and the corresponding adjusted games (S, v_a^N) and $(N\setminus S, v_a^N)$. Continue taking binary partitions and adjusted games for every non-singleton coalition until only singletons remain. At the end of the process, every player $i \in N$ gets a share of $v(N)$, and this amount can be expressed explicitly as the evaluation of a composition of adjusted games. The described procedure can be applied for every binary total partition of N. We study the solution given by the average of all the allocations obtained with all the binary total partitions of N. If $\eta_i^P(N, v)$ is the amount obtained by $i \in N$ with $P \in B_N$ then the proposed solution for i is

$$\eta_i(N, v) = \frac{1}{p(n)}\sum_{P \in B_N}\eta_i^P(N, v)\,. \tag{5}$$

Example 3.1. Let $N = \{1, 2, 3, 4\}$ and $(N.v) \in G^4$. If

$$P = \{\,\{1,2,3,4\}, \{\{1,4\}, \{2,3\}\}, \{\{1\}, \{4\}, \{2\}, \{3\}\}\,\}$$

then the amount obtained by each player with P is given by the coordinates of the allocation

$$\left(\ ((v_a^N)_a^{\{1,4\}})(1),\ ((v_a^N)_a^{\{2,3\}})(2),\ ((v_a^N)_a^{\{2,3\}})(3),\ ((v_a^N)_a^{\{1,4\}})(4)\ \right)$$

To write each amount, one considers the nested coalitions from N to $\{i\}$ and to compute each amount by explicitly using Equation (2) and proceeding similarly as in example 2.1.

The solution —Equation (5)— satisfies a recursive formula. If $\{S, N\setminus S\}$ is a binary partition of N then $i \in S$ gets $\eta_i^P(S, v_a^N)$ with $P \in B_S$. If one fixes S then i obtains $\eta_i^P(S, v_a^N)$ for every binary total partition of $N\setminus S$, i.e., $p(n-s)$ times. Thus

$$\eta_i(N, v) = \frac{1}{p(n)}\sum_{P \in B_N}\eta_i^P(N, v)$$

$$
\begin{aligned}
&= \frac{1}{p(n)} \sum_{\substack{s \ni i \\ S \neq N}} \sum_{P \in B_S} p(n-s) \eta_i^P(S, v_a^N) \\
&= \frac{1}{p(n)} \sum_{\substack{s \ni i \\ S \neq N}} p(s) p(n-s) \eta_i(S, v_a^N) \,.
\end{aligned} \tag{6}
$$

Example 3.2. With the change of the respective letters by numbers, we model the cooperative situation of example 1.1 with the game $v(1) = v(2) = 1/4$, $v(3) = 0$, $v(12) = 3/4$, $v(13) = v(23) = 1/2$ and $v(123) = 1$. By using Equations (1) and (2) or (6), we have

$$
\begin{aligned}
\eta_1(N, v) &= \frac{1}{3}\left[v_a^N(1) + (v_a^N)_a^{\{1,2\}}(1) + (v_a^N)_a^{\{1,3\}}(1)\right] \\
&= \frac{5}{12}
\end{aligned}
$$

Similarly, we obtain $\eta_2(N, v) = 5/12$ and $\eta_3(N, v) = 2/12$. The allocation obtained is the Shapley value of the game (N, v).

The solution of Equation (5) turns out to be the Shapley value, and the recursive formula of Equation (6) characterizes it.

Theorem 3.1. The solution φ satisfies the recursive formula

$$
\varphi_i(N, v) = \frac{1}{p(n)} \sum_{\substack{s \ni i \\ S \neq N}} p(s) p(n-s) \varphi_i(S, v_a^N) \,. \tag{7}
$$

and the condition $\varphi_i(\{i\}, v) = v(i)$ if and only if φ is the Shapley value.

Proof. Let $Sh_i(N, v)$ be the Shapley value of i in the game $(N, v) \epsilon G^n$. Since Equation (7) is a recursive formula, it is enough to show that the Shapley value satisfies this formula for every size of the set of players $n \geq 2$. For $n = 2$ the result readily follows since the Shapley value and the solution η coincide with the standard solution for 2-player games. If the Shapley value satisfies Equation (7) for every size of the set of players less than n then Equation (7) is valid with $\varphi_i(S, v_a^N)$ replaced by $Sh_i(S, v_a^N)$. It remains to show that the resulting expression is $Sh_i(N, v)$.

From the additivity of the Shapley value and the definition of the adjusted game (S, v_a^N) we have

$$
\eta_i(N, v) = \frac{1}{2p(n)} \sum_{\substack{s \ni i \\ S \neq N}} p(s) p(n-s) \big(Sh_i(S, v) + Sh_i(S, v^*)\big) \,,
$$

where $Sh_i(S, v^*)$ is the Shapley value of i in the dual game of v restricted to S, i.e. $v^*(T) = v(N) - v(N \setminus T),\ T \subseteq S$. Explicitly

$$
\begin{aligned}
\eta_i(N, v) &= \sum_{\substack{s \ni i \\ S \neq N}} \frac{p(s)p(n-s)}{2p(n)} \left[\sum_{T \not\ni i} \frac{t!\,(s-t-1)!}{s!} \big(v(T \cup i) - v(T) + v^*(T \cup i) - v^*(T)\big)\right] \\
&= \sum_{T \not\ni i} \sum_{\substack{S \ni i \\ S \neq N}} \frac{p(s)p(n-s)}{2p(n)} \frac{t!\,(s-t-1)!}{s!} \big(v(T \cup i) - v(T) + v^*(T \cup i) - v^*(T)\big)
\end{aligned}
$$

$$= \sum_{T \not\ni i} \left(\sum_{s=t+1}^{n-1} \frac{p(s)p(n-s)}{2p(n)} \frac{t!\,(s-t-1)!}{s!} \binom{n-t-1}{s-t-1} \right)$$
$$\cdot \big(v(T \cup i) - v(T) + v^*(T \cup i) - v^*(T)\big)$$
$$= \sum_{T \not\ni i} \frac{t!\,(n-t-1)!}{2p(n) \cdot n!} \left[\sum_{s=t+1}^{n-1} p(s)p(n-s) \binom{n}{s} \big(v(T \cup i) - v(T) + v^*(T \cup i) - v^*(T)\big) \right]. \quad (8)$$

The term $p(s)p(n-s)\binom{n}{s}$ does not change if we interchange s and $n-s$. Thus putting $\tilde{T} = N \setminus (T \cup i)$ we obtain

$$\sum_{s=t+1}^{n-1} p(s)p(n-s)\binom{n}{s}\big(v^*(T \cup i) - v^*(T)\big) = \sum_{s=1}^{n-t-1} p(s)p(n-s)\binom{n}{s}\Big(v(N \setminus T) - v\big(N \setminus (T \cup i)\big)\Big).$$
$$= \sum_{s=1}^{n-t-1} p(s)p(n-s)\binom{n}{s}\Big(v\big(\tilde{T} \cup i\big) - v\big(\tilde{T}\big)\Big) \quad (9)$$

For coalitions not containing i, the coalitions of size t are in one-to-one correspondence with the coalitions of size $n-t-1$. Thus, we can change $n-t-1$ by t and $\tilde{T}$ by an appropriate T in the last sum of Equation (9). Substituting in Equation (8) we obtain, using Equation (4),

$$\eta_i(N,v) = \sum_{T \not\ni i} \frac{t!\,(n-t-1)!}{2p(n) \cdot n!} \sum_{s=1}^{n-1} \binom{n}{s} p(s)p(n-s)\big(v(T \cup i) - v(T)\big)$$
$$= \sum_{T \not\ni i} \frac{t!\,(n-t-1)!}{n!}\big(v(T \cup i) - v(T)\big)$$
$$= Sh_i(N,v)$$

4 Conclusions

We have introduced a procedure to extend the standard solution to solve 2-player cooperative games. This procedure is based on the recursive application of a simple principle of sharing an estate between two nonempty disjoint coalitions that are part of a cooperative game: every coalition gets its own worth, and the surplus is equally divided between both coalitions. This procedure is very simple to understand and allows us to obtain a solution for n-player cooperative games in a direct way without the need to introduce a reduced game. The solution obtained turns out to be the Shapley value, and the recursion technique allows us to establish the recursive formula of Equation (7).

We have shown that a solution φ satisfies Equation (7) if and only if φ is the Shapley value, thus characterizing the Shapley value as the unique solution concept which satisfies this recursive formula. This formula can be used to implement the Shapley value computationally in an easy way since all its components can be computed recursively.

We conclude by noticing that the whole binary recursive procedure can be applied to obtain new solution concepts, which extend solutions to solve 2-player cooperative games, using adjusted games and the set of binary total partitions.

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