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Optimization for the design of reinforced concrete corner combined footings

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Abstract. This paper presents the simplified and generalized equations to estimate the optimal design based in the concept of minimum cost for the reinforced concrete corner combined footings under axial load and biaxial moments in each column that considers the linear pressure of the soil acting on the footing contact surface. This work is presented in two stages: in the first stage the minimum contact surface on the footing is obtained, and in the second stage the minimum cost for design is obtained. The formulation was developed under the condition that the derivative of the moment is the shear force. Four examples are shown to obtain the minimum cost for the complete design. The solution is obtained with the help of Maple-15 software that solves these types of problems. The results show that there is no direct relationship between the optimal area and the minimum cost design.
Keywords: optimization, corner combined footing.

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1 Introduction

Foundations or footings are the main elements for the construction of buildings and bridges, which serve to transmit the loads of the superstructure to the supporting ground.

Foundations can be classified as:

1. Shallow foundations (strip footing, isolated footing, combined footing, strap or cantilever footing, raft or slab foundations) lightweight structures and/or high load capacity of the soil.
2. Deep foundations (foundation piles, foundation pits or caissons) heavy constructions and/or shallow soils with low load capacity.

Structural engineers usually use trial and error approaches to address with design problems when they need to obtain the most economical design of a structural element in terms of its material cost, meeting all the safety requirements imposed by the design codes.

The optimal design of structures has been the subject of many studies in the field of structural design. The goal of a designer is to develop a “best solution” for structural design under certain considerations. An optimal solution usually involves the most economical structure without impairing the functional purposes of the structure.

The main contributions of various researchers on the subject of optimization and mathematical models for the design for reinforced concrete foundations are: Algin formulated a practical algebraically solution to obtain the minimum area of a rectangular isolated footing subjected to a vertical load and moments in both axes (biaxial bending) [1]. Wang proposed a design approach that integrates economic design optimization with reliability-based methodologies to assess the ultimate and serviceability limit state requirements to rationally account for geotechnical-related uncertainties [2]. Smith-Pardo developed some design aids by graphics to obtain the restriction effect in the bases of columns and walls supported on shallow foundations [3]. Basudhar et al. investigated the optimal cost analysis and design for a circular footing subjected to generalized loads employing the sequential unconstrained minimization technique in conjunction with Powell’s conjugate direction method for multidimensional search and quadratic interpolation method for a dimensional minimization [4]. Al-Ansari proposed an analytical model to obtain the cost of an optimal design of reinforced concrete isolated footings with yield strength of reinforcing steel bars and compression strength of concrete based in shear and flexural capacity of the footing [5]. Imanzadeh et al. studied two approaches for the design of continuous spread footings, the first design using a one-dimensional finite element model and the second design using a three-dimensional finite element model [6]. Ukritchon and Keawsawasvong presented a practical model for the optimal design of a continuous footing under to vertical and horizontal loads, to obtain the minimum footing size and the minimum reinforcement steel, and it is formulated in a non-linear minimization form [7]. López-Chavarría et al. studied the optimal dimensioning for the corner combined footings to obtain the most economical contact surface with the ground (optimal area), which supports an axial load, and two orthogonal moments around of the X and Y axes by each column [8]. Luévanos-Rojas et al. developed an optimal design for rectangular isolated footings using the linear soil pressure, also numerical examples are presented to estimate the minimum cost design of the materials used for the building of the footings supporting an axial load, a moment around of the X axis, and other moment around of the Y axis in accordance to the building code (ACI 318-13) [9]. Yeh and Huang studied the optimization of reinforced concrete isolated footings using genetic algorithms, and also investigated the effects of the yield strength of steel, the compressive strength of concrete, the eccentricity of the axial load and the steel bar size [10]. Velázquez-Santillán et al. investigated the optimal design model for reinforced concrete rectangular combined footings to obtain the minimum cost design in accordance with the building code (ACI 318-14) [11]. Rawat and Mital described a simplified approach for the design of reinforced concrete isolated footings with eccentric load that explicitly considers the structural requirements and economics simultaneously, and therefore, results give a foundation with minimum cost [12]. Luévanos-Rojas et al. proposed an optimal model to obtain the minimum dimensions (part 1) and a mathematical model to obtain the thickness and the reinforcing steel (part 2) for the T-shaped combined footings [13, 14]. Islam and Rokonuzzaman introduced an optimal design process (construction cost) for shallow isolated column footing in sands using genetic algorithms that include the design parameters and design requirements as constraints [15]. Nigdeli et al. developed a methodology based in metaheuristic to obtain the optimal cost of reinforced concrete footings using several classical algorithms that are powerful to deal with non-linear optimization problems [16]. Aguilera-Mancilla et al. and Yáñez-Palafox et al. developed an optimal model to obtain the minimum dimensions (part 1) and a mathematical model to obtain the thickness and the reinforcing steel (part 2) for the strap combined footings, respectively [17, 18]. López-Chavarría et al. investigated the optimal design for reinforced concrete circular isolated footings based on a criterion of minimum cost in accordance with the building code (ACI 318-14) [19]. Farías-Montemayor et al. investigated an optimized model to obtain the minimum dimensions (part 1) and an optimal model to obtain the thickness and the reinforcing steel based on a criterion of minimum cost (part 2) for the rectangular pile caps supported on a group of piles [20, 21]. Luévanos-Rojas et al. obtained a mathematical model to obtain the thickness and the reinforcing steel for the design of corner combined footings [22]. Solorzano and Plevris presented the design of reinforced concrete rectangular-shaped isolated footings using the genetic algorithm in accordance with the American Concrete Institute ACI 318-19 [23]. Pane et al. used an approximate numerical model to evaluate the actions in the foundation ground and in the tie-beams in terms of foundation size and cost, considering the capacity of tie-beams to absorb part of the bending moments, which are generally attributed only to the foundations [24]. Galvis and Smith-Pardo presented design aids, experimental verification, and examples for rectangular and circular shallow foundations subjected to axial load and biaxial moment [25].

According to the researched literature, the documents closest to the topic being addressed are: 1) Optimal dimensioning for combined corner footings [8], but equations are presented in a very specific way, without showing the different shapes or limitations that the footing may have; 2) An analytical model for the design of corner combined footings [22], but they present only the equations for the design, without showing the optimal design or minimum cost of the footing.

This paper shows two optimal models for the design of reinforced concrete corner combined footings, the first model presents the simplified and generalized equations to obtain the minimum area of contact on the ground and the different shapes or limitations that the footing may have, the second model shows the simplified and generalized equations to estimate the optimal design or minimum cost with the design parameters and the constraint functions in accordance with the building code requirements for structural concrete of the American Concrete Institute. Also, four practical examples for design are presented: first - unconstrained sides, second - constraint in the X direction, third - constraint in the Y direction, fourth - constraints in the X and Y directions. The solution is obtained with the help of any software that solves these types of problems.

2 Formulation of the optimal model

Fig. 1 shows a corner combined footing that supports three rectangular columns of different dimensions (a corner column and two inner columns with an boundary) under an axial load and two orthogonal moments (bidirectional bending) in each column.

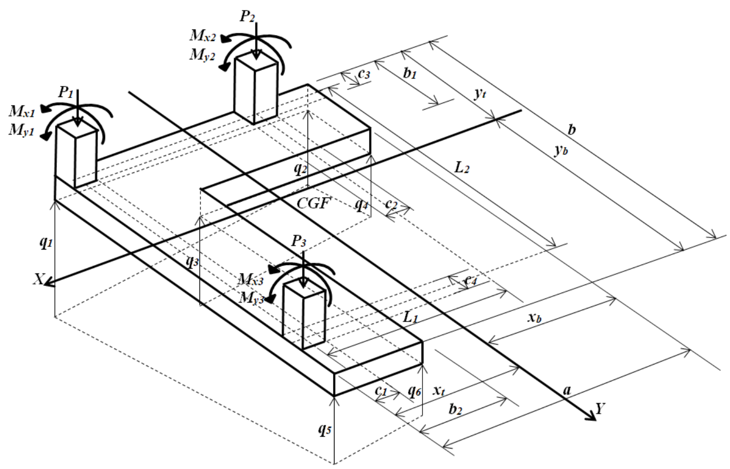

Fig. 1. Corner combined footing.

Table 1 shows the coordinates of the pressures below footing at each vertex.

Table 1. Coordinates of the pressures below of the corner combined footing.

Pressures	q_1	q_2	q_3	q_4	q_5	q_6
Coordinates	x_1	x_2	x_3	x_4	x_5	x_6
	x_t	$x_t - a$	$x_t - b_2$	$x_t - a$	x_t	$x_t - b_2$
	y_1	y_2	y_3	y_4	y_5	y_6
	y_t	y_t	$y_t - b_1$	$y_t - b_1$	$y_t - b$	$y_t - b$

2.1. Model of the minimum contact surface on the ground for the corner combined footings

The objective function to obtain the contact minimum surface on the soil "A_{min}" is [8]:

$$A_{min} = (a - b_2)b_1 + bb_2 \tag{1}$$

The constraint functions are:

$$q_n = \frac{R}{A} + \frac{M_{xT}y_n}{I_x} + \frac{M_{yT}x_n}{I_y} \tag{2}$$

$$R = P_1 + P_2 + P_3 \tag{3}$$

$$M_{xT} = Ry_t + M_{x1} + M_{x2} + M_{x3} - \frac{Rc_3}{2} - P_3L_2 \tag{4}$$

$$M_{yT} = Rx_t + M_{y1} + M_{y2} + M_{y3} - \frac{Rc_1}{2} - P_2L_1 \tag{5}$$

$$y_t = \frac{(a - b_2)b_1{}^2 + b^2b_2}{2[(a - b_2)b_1 + bb_2]} \tag{6}$$

$$y_b = \frac{(2b - b_1)(a - b_2)b_1 + b^2b_2}{2[(a - b_2)b_1 + bb_2]} \tag{7}$$

$$x_t = \frac{a^2b_1 + (b - b_1)b_2{}^2}{2[(a - b_2)b_1 + bb_2]} \tag{8}$$

$$x_b = \frac{a^2b_1 + (2a - b_2)(b - b_1)b_2}{2[(a - b_2)b_1 + bb_2]} \tag{9}$$

$$I_x = \frac{a^2b_1{}^4 + 2ab_1b_2(b - b_1)(2b^2 - bb_1 + b_1{}^2) + b_2{}^2(b - b_1)^4}{12[(a - b_2)b_1 + bb_2]} \tag{10}$$

$$I_y = \frac{b^2b_2{}^4 + 2bb_1b_2(a - b_2)(2a^2 - ab_2 + b_2{}^2) + b_1{}^2(a - b_2)^4}{12[(a - b_2)b_1 + bb_2]} \tag{11}$$

$$0 \leq \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \leq q_{aa} \tag{12}$$

$$y_t + y_b = b \tag{13}$$

$$x_t + x_b = a \tag{14}$$

where: R = Resultant force (kN); M_{xT} = Resultant moment around the X axis (kN-m); M_{yT} = Resultant moment around the Y axis (kN-m); x_n = Distance in the X direction measured from the Y axis to the fiber under study (m); y_n = Distance in the Y direction measured from the X axis to the fiber under study (m); I_x = Moment of inertia around the X axis (m^4); I_y = Moment of inertia around the Y axis (m^4), q_{aa} = Available permissible load capacity of the soil (kN/m^2).

The constraint functions for the geometric conditions are:
The equations for the unconstrained sides are:

$$\begin{gathered} \frac{c_1}{2} + L_1 + \frac{c_2}{2} \leq a \\ \frac{c_3}{2} + L_2 + \frac{c_4}{2} \leq b \end{gathered} \tag{15}$$

The equations for a constraint in the X direction are:

$$\begin{gathered} \frac{c_1}{2} + L_1 + \frac{c_2}{2} = a \\ \frac{c_3}{2} + L_2 + \frac{c_4}{2} \leq b \end{gathered} \tag{16}$$

The equations for a constraint in the Y direction are:

$$\begin{gathered} \frac{c_1}{2} + L_1 + \frac{c_2}{2} \leq a \\ \frac{c_3}{2} + L_2 + \frac{c_4}{2} = b \end{gathered} \tag{17}$$

The equations for two constraints in the X and Y directions are:

$$\begin{gathered} \frac{c_1}{2} + L_1 + \frac{c_2}{2} = a \\ \frac{c_3}{2} + L_2 + \frac{c_4}{2} = b \end{gathered} \tag{18}$$

2.2. Model of minimum cost for design of corner combined footings

2.2.1. Equations for the bending shear and bending moments

The critical sections for factored moments according to the ACI code are presented on the axes: *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i* and *j* (see Fig. 2).

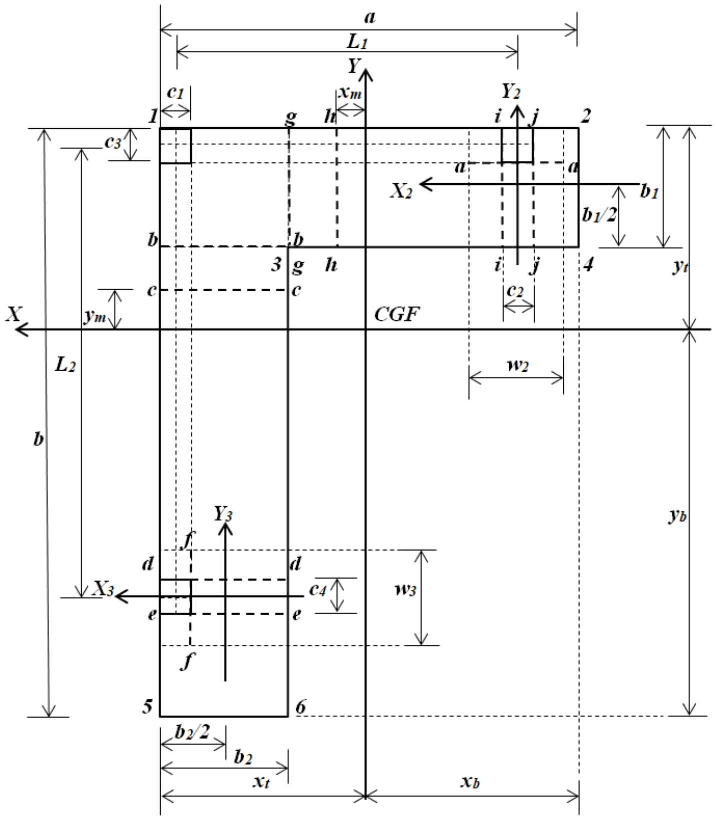

Fig. 2. Moments

The critical sections for factored bending shear according to the ACI code are presented on the axes: *k*, *l*, *m*, *n*, *o*, *p*, *q* and *r* (see Fig. 3).

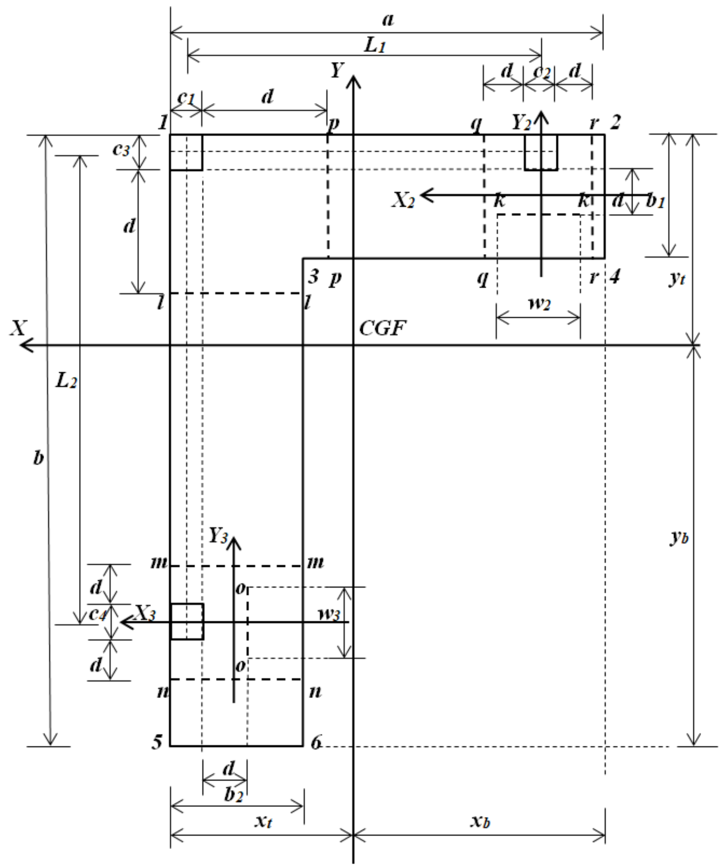

Fig. 3. Bending shear

The factored bending shear and the factored moment acting on the footing in the X_2 axis for the interval $-b_1/2 \leq y_2 \leq b_1/2 - c_3/2$ [22]:

$$V_{uy_2} = -\frac{P_{u2}\left[12(b_1 - c_3)y_2^{\,2} + b_1^{\,2}(4y_2 - b_1 + 3c_3)\right]}{4b_1^{\,3}} - \frac{3M_{ux2}\left(4y_2^{\,2} - b_1^{\,2}\right)}{2b_1^{\,3}} \tag{19}$$

$$M_{ux_2} = -\frac{P_{u2}y_2\left[4(b_1 - c_3)y_2^{\,2} + 2b_1^{\,2}y_2 - b_1^{\,2}(b_1 - 3c_3)\right]}{4b_1^{\,3}} - \frac{M_{ux2}y_2\left(4y_2^{\,2} - 3b_1^{\,2}\right)}{2b_1^{\,3}} + \frac{P_{u2}(b_1 - 2c_3) + 4M_{ux2}}{8} \tag{20}$$

where: the analysis width on the X_2 axis is: $w_2 = c_2 + d/2$ for limit column in the X_2 direction, and $w_2 = c_2 + d$ for the column without limit.

Now, substituting $y_2 = b_1/2 - c_3 - d$ into Eq. (19) the bending shear V_{uk} that acts on the k axis is obtained, and substituting $y_2 = b_1/2 - c_3$ into Eq. (20) the moment M_{ua} that acts on the a axis is obtained.

The factored bending shear and factored moment acting on the footing in the X axis for the interval $y_t - c_3/2 \leq y \leq y_t$ [22]:

$$V_{uy} = -\frac{R_u a(y_t - y)}{A} - \frac{M_{uxT}a(y_t^{\,2} - y^2)}{2I_x} - \frac{M_{uyT}a(2x_t - a)(y_t - y)}{2I_y} \tag{21}$$

$$M_{ux} = \frac{R_u a(y_t - y)^2}{2A} + \frac{M_{uxT}a(2y_t^{\,3} - 3y_t^{\,2}y + y^3)}{6I_x} \tag{22}$$

where: the analysis width on the X axis is a for this interval.

The factored bending shear and factored moment acting on the footing in the X axis for the interval $y_t - b_1 \leq y \leq y_t - c_3/2$ [22]:

$$V_{uy} = P_{u1} + P_{u2} - \frac{R_u a(y_t - y)}{A} - \frac{M_{uxT}a(y_t^{\,2} - y^2)}{2I_x} - \frac{M_{uyT}a(2x_t - a)(y_t - y)}{2I_y} \tag{23}$$

$$M_{ux} = \frac{R_u a(y_t - y)^2}{2A} + \frac{M_{uxT} a(2{y_t}^3 - 3{y_t}^2 y + y^3)}{6I_x} - (P_{u1} + P_{u2})\left(y_t - y - \frac{c_3}{2}\right) - M_{ux1} - M_{ux2} \quad (24)$$

where: the analysis width on the X axis is a for this interval.

Now, substituting $y = y_t - c_3 - d$ into Eq. (23) (if the l axis falls within of this interval) the bending shear V_{ul} is obtained, and substituting $y = y_t - b_1$ into Eq. (24) the moment M_{ub} that acts on the b axis is obtained.

The factored bending shear and factored moment acting on the footing in the X axis for the interval $y_t - L_2 - c_3/2 \leq y \leq y_t - b_1$ [22]:

$$V_{uy} = -\frac{R_u[ab_1 + b_2(y_t - y - b_1)]}{A} - \frac{M_{uxT}\{ab_1(2y_t - b_1) + b_2[(y_t - b_1)^2 - y^2]\}}{2I_x} - \frac{M_{uyT}[ab_1(2x_t - a) + b_2(2x_t - b_2)(y_t - y - b_1)]}{2I_y} + P_{u1} + P_{u2} \quad (25)$$

$$M_{ux} = \frac{R_u[ab_1(2y_t - 2y - b_1) + b_2(y_t - y - b_1)^2]}{2A} + \frac{M_{uxT} ab_1[2(3{y_t}^2 - 3y_t b_1 + {b_1}^2) - 3y(2y_t - b_1)]}{6I_x} + \frac{M_{uxT} b_2[y^3 + (y_t - b_1)^2(2y_t - 3y - 2b_1)]}{6I_x} - (P_{u1} + P_{u2})\left(y_t - y - \frac{c_3}{2}\right) - M_{ux1} - M_{ux2} \quad (26)$$

where: the analysis width on the X axis is b_2 for this interval.

Now, substituting $y = y_t - c_3 - d$ into Eq. (25) (if the l axis falls within of this interval) the bending shear V_{ul} is obtained, and substituting $y = y_t - c_3/2 - L_2 + c_4/2 + d$ into Eq. (25) the bending shear V_{um} that acts on the m axis is obtained. Now, substituting $y = y_t - b_1$ into Eq. (26) the moment M_{ub} that acts on the b axis is obtained, Eq. (25) is set equal to zero to obtain the position of the maximum moment y_m and later it is substituted into Eq. (26) and the maximum moment M_{uc} is obtained, and substituting $y = y_t - c_3/2 - L_2 + c_4/2$ into Eq. (26) the moment M_{ud} that acts on the d axis is obtained.

The factored bending shear and factored moment acting on the footing in the X axis for the interval $y_t - b \leq y \leq y_t - L_2 - c_3/2$ [22]:

$$V_{uy} = -\frac{R_u[ab_1 + b_2(y_t - y - b_1)]}{A} - \frac{M_{uyT}[ab_1(2x_t - a) + b_2(2x_t - b_2)(y_t - y - b_1)]}{2I_y} - \frac{M_{uxT}\{ab_1(2y_t - b_1) + b_2[(y_t - b_1)^2 - y^2]\}}{2I_x} + P_{u1} + P_{u2} + P_{u3} \quad (27)$$

$$M_{ux} = \frac{R_u[ab_1(2y_t - 2y - b_1) + b_2(y_t - y - b_1)^2]}{2A} + \frac{M_{uxT} ab_1[2(3{y_t}^2 - 3y_t b_1 + {b_1}^2) - 3y(2y_t - b_1)]}{6I_x} + \frac{M_{uxT} b_2[y^3 + (y_t - b_1)^2(2y_t - 3y - 2b_1)]}{6I_x} - R_u\left(y_t - y - \frac{c_3}{2}\right) + P_{u3}L_2 - M_{ux1} - M_{ux2} - M_{ux3} \quad (28)$$

where: the analysis width on the X axis is b_2 for this interval.

Now, substituting $y = y_t - c_3/2 - L_2 - c_4/2 - d$ into Eq. (27) the bending shear V_{un} that acts on the n axis is obtained. Now, substituting $y = y_t - c_3/2 - L_2 - c_4/2$ into Eq. (28) the moment M_{ue} that acts on the e axis is obtained.

The factored bending shear and factored moment acting on the footing in the Y_3 axis for the interval $-b_2/2 \leq x_3 \leq b_2/2 - c_1/2$ [22]:

$$V_{ux_3} = -\frac{P_{u3}[12(b_2 - c_1){x_3}^2 + {b_2}^2(4x_3 - b_2 + 3c_1)]}{4{b_2}^3} - \frac{3M_{uy3}(4{x_3}^2 - {b_2}^2)}{2{b_2}^3} \quad (29)$$

$$M_{uy_3} = -\frac{P_{u3}x_3[4(b_2 - c_1){x_3}^2 + 2{b_2}^2 x_3 - {b_2}^2(b_2 - 3c_1)]}{4{b_2}^3} - \frac{M_{uy3}x_3(4{x_3}^2 - 3{b_2}^2)}{2{b_2}^3} + \frac{P_{u3}(b_2 - 2c_1) + 4M_{uy3}}{8} \quad (30)$$

where: the analysis width on the Y_3 axis is: $w_3 = c_4 + d/2$ for limit column in the Y_3 direction, and $w_3 = c_4 + d$ for the column without limit.

Now, substituting $x_3 = b_2/2 - c_1 - d$ into Eq. (29) the bending shear V_{uo} that acts on the o axis is obtained, and substituting $x_3 = b_2/2 - c_1$ into Eq. (30) the moment M_{uf} that acts on the f axis is obtained.

The factored bending shear and factored moment acting on the footing in the Y axis for the interval $x_t - c_1/2 \leq x \leq x_t$ [22]:

$$V_{ux} = -\frac{R_u b(x_t - x)}{A} - \frac{M_{uxT} b(2y_t - b)(x_t - x)}{2I_x} - \frac{M_{uyT} b(x_t{}^2 - x^2)}{2I_y} \quad (31)$$

$$M_{uy} = \frac{R_u b(x_t - x)^2}{2A} + \frac{M_{uyT} b(2x_t{}^3 - 3x_t{}^2 x + x^3)}{6I_y} \quad (32)$$

where: the analysis width on the Y axis is b for this interval.

The factored bending shear and factored moment acting on the footing in the Y axis for the interval $x_t - b_2 \leq x \leq x_t - c_1/2$ [22]:

$$V_{ux} = P_{u1} + P_{u3} - \frac{R_u b(x_t - x)}{A} - \frac{M_{uxT} b(2y_t - b)(x_t - x)}{2I_x} - \frac{M_{uyT} b(x_t{}^2 - x^2)}{2I_y} \quad (33)$$

$$M_{uy} = \frac{R_u b(x_t - x)^2}{2A} + \frac{M_{uyT} b(2x_t{}^3 - 3x_t{}^2 x + x^3)}{6I_y} - (P_{u1} + P_{u3})\left(x_t - x - \frac{c_1}{2}\right) - M_{uy1} - M_{uy3} \quad (34)$$

where: the analysis width on the Y axis is b for this interval.

Now, substituting $x = x_t - c_1 - d$ into Eq. (33) (if the p axis falls within of this interval) the bending shear V_{up} is obtained, and substituting $x = x_t - b_2$ into Eq. (34) the moment M_{ug} that acts on the g axis is obtained.

The factored bending shear and factored moment acting on the footing in the Y axis for the interval $x_t - L_1 - c_1/2 \leq x \leq x_t - b_2$ [22]:

$$V_{ux} = -\frac{R_u[bb_2 + b_1(x_t - x - b_2)]}{A} - \frac{M_{uxT}[bb_2(2y_t - b) + b_1(2y_t - b_1)(x_t - x - b_2)]}{2I_x} - \frac{M_{uyT}\{bb_2(2x_t - b_2) + b_1[(x_t - b_2)^2 - x^2]\}}{2I_y} + P_{u1} + P_{u3} \quad (35)$$

$$M_{uy} = \frac{R_u[bb_2(2x_t - 2x - b_2) + b_1(x_t - x - b_2)^2]}{2A} + \frac{M_{uyT} bb_2[2(3x_t{}^2 - 3x_t b_2 + b_2{}^2) - 3x(2x_t - b_2)]}{6I_y} + \frac{M_{uyT} b_1[x^3 + (x_t - b_2)^2(2x_t - 3x - 2b_2)]}{6I_y} - (P_{u1} + P_{u3})\left(x_t - x - \frac{c_1}{2}\right) - M_{uy1} - M_{uy3} \quad (36)$$

where: the analysis width on the Y axis is b_2 for this interval.

Now, substituting $x = x_t - c_1 - d$ into Eq. (35) (if the p axis falls within of this interval) the bending shear V_{up} is obtained, and substituting $x = x_t - c_1/2 - L_1 + c_2/2 + d$ into Eq. (35) the bending shear V_{uq} that acts on the q axis is obtained. Now, substituting $x = x_t - b_2$ into Eq. (36) the moment M_{ug} that acts on the g axis is obtained, Eq. (35) is set equal to zero to obtain the position of the maximum moment x_m and later it is substituted into Eq. (36) and the maximum moment M_{uh} is obtained, and substituting $x = x_t - c_1/2 - L_1 + c_2/2$ into Eq. (36) the moment M_{ui} that acts on the i axis is obtained.

The factored bending shear and factored moment acting on the footing in the Y axis for the interval $x_t - a \leq x \leq x_t - L_1 - c_1/2$ [22]:

$$V_{ux} = -\frac{R_u[bb_2 + b_1(x_t - x - b_2)]}{A} - \frac{M_{uxT}[bb_2(2y_t - b) + b_1(2y_t - b_1)(x_t - x - b_2)]}{2I_x} - \frac{M_{uyT}\{bb_2(2x_t - b_2) + b_1[(x_t - b_2)^2 - x^2]\}}{2I_y} + P_{u1} + P_{u2} + P_{u3} \quad (37)$$

$$M_{uy} = \frac{R_u[bb_2(2x_t - 2x - b_2) + b_1(x_t - x - b_2)^2]}{2A} + \frac{M_{uyT} bb_2[2(3x_t{}^2 - 3x_t b_2 + b_2{}^2) - 3x(2x_t - b_2)]}{6I_y} + \frac{M_{uyT} b_1[x^3 + (x_t - b_2)^2(2x_t - 3x - 2b_2)]}{6I_y} - R_u\left(x_t - x - \frac{c_1}{2}\right) + P_{u2}L_1 - M_{uy1} - M_{uy2} - M_{uy3} \quad (38)$$

where: the analysis width on the Y axis is b_2 for this interval.

Now, substituting $x = x_t - c_1/2 - L_1 - c_2/2 - d$ into Eq. (37) the bending shear V_{ur} that acts on the r axis is obtained. Now, substituting $x = x_t - c_1/2 - L_1 - c_2/2$ into Eq. (38) the moment M_{uj} that acts on the j axis is obtained.

2.2.2. Equations for the punching shear

The critical sections for the factored punching shear according to the ACI code are presented on the perimeter formed by points 1, 7, 8 and 9 in column 1, by points 10, 11, 12 and 13 in column 2, and by points 14, 15, 16 and 17 in column 3 (see Fig. 4).

$$V_{up1} = P_{u1} - \frac{R_u(c_1 + d/2)(c_3 + d/2)}{A} - \frac{M_{uxT}(2y_t - c_3 - d/2)(c_1 + d/2)(c_3 + d/2)}{2I_x} - \frac{M_{uyT}(2x_t - c_1 - d/2)(c_1 + d/2)(c_3 + d/2)}{2I_y} \quad (39)$$

For limit column in the X_2 direction:

$$V_{up2} = P_{u2} - \frac{R_u(c_2 + d/2)(c_3 + d/2)}{A} - \frac{M_{uxT}(2y_t - c_3 - d/2)(c_2 + d/2)(c_3 + d/2)}{2I_x} - \frac{M_{uyT}(2x_t - 2L_1 - c_1 + d/2)(c_2 + d/2)(c_3 + d/2)}{2I_y} \quad (40)$$

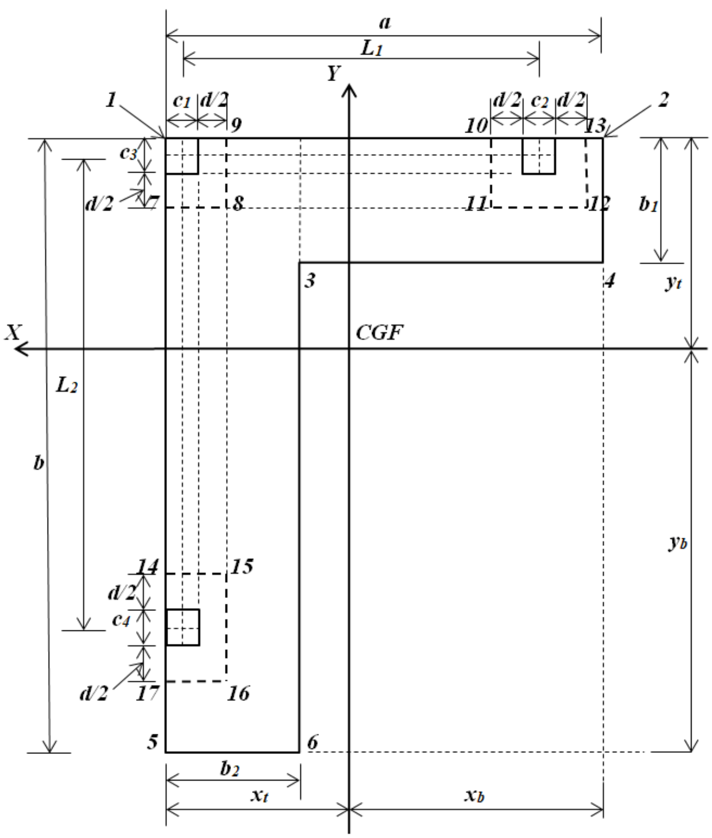

Fig. 4. Punching shear

For the column without limit:

$$V_{up2} = P_{u2} - \frac{R_u(c_2 + d)(c_3 + d/2)}{A} - \frac{M_{uxT}(2y_t - c_3 - d/2)(c_2 + d)(c_3 + d/2)}{2I_x} - \frac{M_{uyT}(2x_t - 2L_1 - c_1)(c_2 + d)(c_3 + d/2)}{2I_y} \quad (41)$$

For limit column in the Y_3 direction:

$$V_{up3} = P_{u3} - \frac{R_u(c_4 + d/2)(c_1 + d/2)}{A} - \frac{M_{uyT}(2x_t - c_1 - d/2)(c_4 + d/2)(c_1 + d/2)}{2I_y} - \frac{M_{uxT}(2y_t - 2L_2 - c_3 + d/2)(c_4 + d/2)(c_1 + d/2)}{2I_x} \quad (42)$$

For the column without limit:

$$V_{up3} = P_{u3} - \frac{R_u(c_4 + d)(c_1 + d/2)}{A} - \frac{M_{uyT}(2x_t - c_1 - d/2)(c_4 + d)(c_1 + d/2)}{2I_y} - \frac{M_{uxT}(2y_t - 2L_2 - c_3)(c_4 + d)(c_1 + d/2)}{2I_x} \quad (43)$$

2.3. Objective function to obtain the minimum cost

The total cost C_T for the corner combined footing is obtained by the following equation:

$$C_T = V_c C_c + V_s \gamma_s C_s \quad (44)$$

where: C_c = cost of concrete for 1 m^3 in dollars, C_s = cost of reinforcing steel for 1 kN of steel in dollars, V_s = volume of reinforcing steel, V_c = volume of concrete, and γ_s = steel density = 76.94 kN/m^3.

The volumes for the corner combined footings are:

$$V_s = (A_{sxTL} + A_{sxBL})a + (A_{syTL} + A_{syBL})b + (A_{sxTT} + A_{sxBT} + A_{sP3})b_2 + (A_{syTT} + A_{syBT} + A_{sP2})b_1 \quad (45)$$

$$V_c = [ab_1 + (b - b_1)b_2]t - (A_{sxTL} + A_{sxBL})a - (A_{syTL} + A_{syBL})b - (A_{sxTT} + A_{sxBT} + A_{sP3})b_2 - (A_{syTT} + A_{syBT} + A_{sP2})b_1 \quad (46)$$

where: t = total thickness of the footing, A_{sxTL} = longitudinal steel area along of the distance "a" at the top with a width "b_1" (X axis direction), A_{sxBL} = longitudinal steel area along of the distance "a" at the bottom with a width "b_1" (X axis direction), A_{syTL} = longitudinal steel area along of the distance "b" at the top with a width "b_2" (Y axis direction), A_{syBL} = longitudinal steel area along of the distance "b" at the bottom with a width "b_2" (Y axis direction), A_{sP3} = steel area at the bottom of the column 3 with a width w_3 (X axis direction), A_{sxTT} = steel area at the top of the surplus b_1 with a width $b - b_1$ (X axis direction), A_{sxBT} = steel area at the bottom of the surplus b_1 and w_3 with a width $b - b_1 - w_3$ (X axis direction), A_{sP2} = steel area at the bottom of the column 2 with a width w_2 (Y axis direction), A_{syTT} = steel area at the top of the surplus b_2 with a width $a - b_2$ (Y axis direction), A_{syBT} = steel area at the bottom of the surplus b_2 and w_2 with a width $a - b_2 - w_2$ (Y axis direction).

Now, substituting Eqs. (45) and (46) into Eq. (44) is shown as equation follows:

$$C_T = C_c\Big[[ab_1 + (b - b_1)b_2]t - (A_{sxTL} + A_{sxBL})a - (A_{syTL} + A_{syBL})b - (A_{sxTT} + A_{sxBT} + A_{sP3})b_2 - (A_{syTT} + A_{syBT} + A_{sP2})b_1\Big] + \gamma_s C_s[(A_{sxTL} + A_{sxBL})a + (A_{syTL} + A_{syBL})b + (A_{sxTT} + A_{sxBT} + A_{sP3})b_2 + (A_{syTT} + A_{syBT} + A_{sP2})b_1] \quad (47)$$

Subsequently, substituting $\alpha = \gamma_s C_s / C_c \rightarrow \gamma_s C_s = \alpha C_c$ into Eq. (47) is presented by the following equation:

$$C_T = C_c\{[(A_{sxTL} + A_{sxBL})a + (A_{syTL} + A_{syBL})b + (A_{sxTT} + A_{sxBT} + A_{sP3})b_2 + (A_{syTT} + A_{syBT} + A_{sP2})b_1](\alpha - 1) + [ab_1 + (b - b_1)b_2]t\} \quad (48)$$

2.4. Constraint functions for the corner combined footings

The constraint for the moment that acts on each section of the footing is [26]:

$$|M_{ua}|, |M_{ub}|, |M_{uc}|, |M_{ud}|, |M_{ue}|, |M_{uf}|, |M_{ug}|, |M_{uh}|, |M_{ui}|, |M_{uj}| \leq \emptyset_f f_y d A_s \left(1 - \frac{0.59 A_s f_y}{b_w d f'_c}\right) \quad (49)$$

where: f_y = Specified yield strength of reinforcement of steel (MPa); f'_c = Specified compressive strength of the concrete at 28 days (MPa); the analysis widths for moment b_w are: for M_{ua} is w_2, for M_{ub}, M_{uc}, M_{ud} and M_{ue} is b_2, for M_{uf} is w_3, for M_{ug}, M_{uh}, M_{ui} and M_{uj} is b_1; the steel areas for moment A_s are: for M_{ua} is A_{sP2}, for M_{ub} is A_{syTLb}, M_{uc} is A_{syTLc}, M_{ud} is A_{syBLd} and M_{ue} is A_{syBLe}, for M_{uf} is A_{sP3}, for M_{ug} is A_{sxTLg}, M_{uh} is A_{sxTLh}, M_{ui} is A_{sxBLi} and M_{uj} is A_{sxBLj}.

The constraint for the bending shear that acts on each section of the footing is [26]:

$$|V_{uk}|, |V_{ul}|, |V_{um}|, |V_{un}|, |V_{uo}|, |V_{up}|, |V_{uq}|, |V_{ur}| \leq 0.17\emptyset_v \sqrt{f'_c}\, b_{ws} d \quad (50)$$

where: the analysis widths for bending shear b_{ws} are: for V_{uk} is w_2, for V_{ul}, V_{um} and V_{un} is b_2, for V_{uo} is w_3, for V_{up}, V_{uq} and V_{ur} is b_1.

The constraint for the punching shear on each section of the footing is [26]:

$$V_{up1}, V_{up2}, V_{up3} \leq \begin{cases} 0.17\emptyset_v\left(1+\dfrac{2}{\beta_c}\right)\sqrt{f'_c}b_0 d \\ 0.083\emptyset_v\left(\dfrac{\alpha_s d}{b_0}+2\right)\sqrt{f'_c}b_0 d \\ 0.33\emptyset_v\sqrt{f'_c}(c_1+c_2+d)d \end{cases} \quad (51)$$

where: the analysis perimeters of the critical section for punching shear b_0 are: for V_{up1} is $c_1 + c_3 + d$ (corner column), for V_{up2} is $2c_3 + c_2 + 2d$ (edge column) and $c_3 + c_2 + d$ (corner column), for V_{up3} is $2c_1 + c_4 + 2d$ (edge column) and $c_1 + c_4 + d$ (corner column); for β_c is ratio of long side to short side of the column; for α_s is 40 for interior column, 30 for edge column, and 20 for corner column.

For the ratios ρ of A_s to $b_w d$ of the footing are [26]:

$$\rho_{P2}, \rho_{yTLb}, \rho_{yTLc}, \rho_{yBLd}, \rho_{yBLe}, \rho_{P3}, \rho_{xTLg}, \rho_{xTLh}, \rho_{xBLi}, \rho_{xBLj} \leq 0.75\left[\frac{0.85\beta_1 f'_c}{f_y}\left(\frac{600}{600+f_y}\right)\right] \quad (52)$$

$$\rho_{P2}, \rho_{yTLb}, \rho_{yTLc}, \rho_{yBLd}, \rho_{yBLe}, \rho_{P3}, \rho_{xTLg}, \rho_{xTLh}, \rho_{xBLi}, \rho_{xBLj} \geq \begin{cases} \dfrac{0.25\sqrt{f'_c}}{f_y} \\ \dfrac{1.4}{f_y} \end{cases} \quad (53)$$

where: ρ_{P2} for M_{ua}, ρ_{yTLb} for M_{ub}, ρ_{yTLc} for M_{uc}, ρ_{yBLd} for M_{ud}, ρ_{yBLe} for M_{ue}, ρ_{P3} for M_{uf}, ρ_{xTLg} for M_{ug}, ρ_{xTLh} for M_{uh}, ρ_{xBLi} for M_{ui}, ρ_{xBLj} for M_{uj}.

For the reinforcing steel areas of the footing are:

$$A_{sP2} = \rho_{P2} w_2 d \quad (54)$$
$$A_{syTLb} = \rho_{yTLb} b_2 d \quad (55)$$
$$A_{syTLc} = \rho_{yTLc} b_2 d \quad (56)$$
$$A_{syBLd} = \rho_{yBLd} b_2 d \quad (57)$$
$$A_{syBLe} = \rho_{yBLe} b_2 d \quad (58)$$
$$A_{sP3} = \rho_{P3} w_3 d \quad (59)$$
$$A_{sxTLg} = \rho_{xTLg} b_1 d \quad (60)$$
$$A_{sxTLh} = \rho_{xTLh} b_1 d \quad (61)$$
$$A_{sxBLi} = \rho_{xBLi} b_1 d \quad (62)$$
$$A_{sxBLj} = \rho_{xBLj} b_1 d \quad (63)$$
$$A_{syTT} = 0.0018(a - b_2)d \quad (64)$$
$$A_{syBT} = 0.0018(a - b_2 - w_2)d \quad (65)$$
$$A_{sxTT} = 0.0018(b - b_1)d \quad (66)$$
$$A_{sxBT} = 0.0018(b - b_1 - w_3)d \quad (67)$$
$$A_{syTL} \geq \begin{cases} A_{syTLb} \\ A_{syTLc} \end{cases} \quad (68)$$
$$A_{syBL} \geq \begin{cases} A_{syBLd} \\ A_{syBLe} \end{cases} \quad (69)$$
$$A_{sxTL} \geq \begin{cases} A_{sxTLg} \\ A_{sxTLh} \end{cases} \quad (70)$$
$$A_{sxBL} \geq \begin{cases} A_{sxBLi} \\ A_{sxBLj} \end{cases} \quad (71)$$

3 Practical examples

Design of a corner combined footing that supports three square columns (see Fig. 1), and the following data is given: the three columns are of 40x40 *cm*; L_1 = 5.00 *m*; L_2 = 6.00 *m*; H = Depth of the footing = 2.0 *m*; P_{D1} = Dead load of the column 1 = 300 *kN*; P_{L1} = Live load of the column 1 = 400 *kN*; M_{Dx1} = Moment around the "X" axis of the dead load of column 1 = 100 *kN-m*; M_{Lx1} = Moment around the "X" axis of the live load of column 1 = 120 *kN-m*; M_{Dy1} = Moment around the "Y" axis of the dead load of column 1 = 130 *kN-m*; M_{Ly1} = Moment around the "Y" axis of the live load of column 1 = 150 *kN-m*; P_{D2} = Dead load of the column 2 = 600 *kN*; P_{L2} = Live load of the column 2 = 800 *kN*; M_{Dx2} = Moment around the “X” axis of the dead load of column 2 = 120 *kN-m*; M_{Lx2} = Moment around the “X” axis of the live load of column 2 = 140 *kN-m*; M_{Dy2} = Moment around the “Y” axis of the dead load of column 2 = 140 *kN-m*; M_{Ly2} = Moment around the “Y” axis of the live load of column 2 = 160 *kN-m*; P_{D3} = Dead load of the column 3 = 800 *kN*; P_{L3} = Live load of the column 3 = 1000 *kN*; M_{Dx3} = Moment around the “X” axis of the dead load of column 3 = 160 *kN-m*; M_{Lx3} = Moment around the “X” axis of the live load of column 3 = 180 *kN-m*; M_{Dy3} = Moment around the “Y” axis of the dead load of column 3 = 180 *kN-m*; M_{Ly3} = Moment around the “Y” axis of the live load of column 3 = 200 *kN-m*; f'_c = 28 *MPa*; f_y = 420 *MPa*; q_a = Allowable load capacity of the soil = 250 kN/m^2; γ_{ppz} = Self-weight of the footing in a cubic meter = 24 kN/m^3; γ_{pps} = Self-weight of soil fill in a cubic meter = 15 kN/m^3. It is assumed that *r* = Coating concrete = 8 *cm*, and α = Relationship between the cost of reinforcing steel and the cost of concrete = 90.

The loads and moments applied to the footing are: P_1 = 700 *kN*; M_{x1} = 220 *kN-m*; M_{y1} = 280 *kN-m*; P_2 = 1400 *kN*; M_{x2} = 260 *kN-m*; M_{y2} = 300 *kN-m*; P_3 = 1800 *kN*; M_{x3} = 340 *kN-m*; M_{y3} = 380 *kN-m*.

The available permissible load capacity of the soil is assumed that is of q_{aa} = 211.00 kN/m^2, because to the available load capacity of the soil is subtracted the self-weight of the footing and the self-weight of soil fill.

Four examples are shown to obtain the minimum cost for the design of reinforced concrete corner combined footings taking into account the same loads and moments applied by each column. Example 1 considers: $c_1/2 + L_1 + c_2/2 \leq a$, $c_3/2 + L_2 + c_4/2 \leq b$, $b_1 \geq 0$, $b_2 \geq 0$ (unconstrained sides). Example 2 takes into account: $c_1/2 + L_1 + c_2/2 = a$, $c_3/2 + L_2 + c_4/2 \leq b$, $b_1 \geq 0$, $b_2 \geq 0$ (constraint in the X direction). Example 3 considers: $c_1/2 + L_1 + c_2/2 \leq a$, $c_3/2 + L_2 + c_4/2 = b$, $b_1 \geq 0$, $b_2 \geq 0$ (constraint in the Y direction). Example 4 takes into account: $c_1/2 + L_1 + c_2/2 = a$, $c_3/2 + L_2 + c_4/2 = b$, $b_1 \geq 0$, $b_2 \geq 0$ (constraints in the X and Y directions).

The solution for the minimum contact surface with the ground by the Maple software is obtained for each example and each example presents the theoretical and practical dimensions (see Table 2) [27].

Table 2. Minimum contact surface with the ground

Concept	Example 1		Example 2		Example 3		Example 4	
	T	P	T	P	T	P	T	P
I_x (m^4)	94.46	99.59	97.39	100.99	73.41	74.55	73.68	74.35
I_y (m^4)	42.37	44.76	41.94	42.92	40.68	32.38	31.44	31.61
M_{xT} (kN-m)	0	144.49	0	147.79	0	-95.91	-128.83	-91.46
M_{yT} (kN-m)	0	154.01	0	59.14	0	-42.62	-97.42	-63.87
R (kN)	3900	3900	3900	3900	3900	3900	3900	3900
a (m)	5.58	5.60	5.40	5.40	6.87	5.50	5.40	5.40
b (m)	7.47	7.50	7.69	7.70	6.40	6.40	6.40	6.40
b_1 (m)	1.57	1.65	1.74	1.80	0.64	1.15	1.20	1.20
b_2 (m)	1.66	1.75	1.53	1.60	2.45	2.50	2.46	2.50
x_t (m)	1.75	1.79	1.75	1.76	1.75	1.74	1.72	1.73
x_b (m)	3.83	3.81	3.65	3.64	5.12	3.76	3.68	3.67
y_t (m)	2.76	2.80	2.76	2.80	2.76	2.73	2.73	2.74
y_b (m)	4.71	4.70	4.93	4.90	3.64	3.67	3.67	3.66
q_1 (kN/m^2)	211.00	210.44	211.00	210.07	211.00	194.71	192.18	193.34
q_2 (kN/m^2)	211.00	191.17	211.00	202.63	211.00	201.95	208.91	204.25
q_3 (kN/m^2)	211.00	202.03	211.00	205.23	211.00	199.48	201.90	199.87
q_4 (kN/m^2)	211.00	188.78	211.00	200.00	211.00	203.43	211.00	205.73
q_5 (kN/m^2)	211.00	199.56	211.00	198.80	211.00	202.94	203.37	201.21
q_6 (kN/m^2)	211.00	193.54	211.00	196.60	211.00	206.23	211.00	206.26
A_{min} (m^2)	18.48	19.48	18.48	19.16	18.48	19.45	19.28	19.48

where: T = Theoretical, P = Practical

The factored loads and the factored moments that act on the corner combined footing due to the columns are: P_{u1} = 1000 *kN*; M_{ux1} = 312 *kN-m*; M_{uy1} = 396 *kN-m*; P_{u2} = 2000 *kN*; M_{ux2} = 368 *kN-m*; M_{uy2} = 424 *kN-m*; P_{u3} = 2560 *kN*; M_{ux3} = 480 *kN-m*; M_{uy3} = 536 *kN-m*.

Now, the practical dimensions of the corner combined footing that supports three square columns are substituted into Eq. (48) to obtain the objective function, and into Eqs. (49) to (71) to obtain the constraint functions.

The minimum cost solution for the design of reinforced concrete corner combined footings by the Maple software is obtained for each example and each example presents the effective depth, the reinforcing steel areas and the percentage of steel (theoretical and practical) (see Table 3) [27].

Table 3. Minimum cost for the design of reinforced concrete corner combined footings

Concept	Example 1		Example 2		Example 3		Example 4	
	T	P	T	P	T	P	T	P
d (cm)	86.91	87.00	92.98	97.00	114.96	117.00	114.20	117.00
ρ_{P2}	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333
ρ_{P3}	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333
ρ_{xBLi}	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333
ρ_{xBLj}	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333
ρ_{xTLg}	0.00723	0.00721	0.00553	0.00505	0.00661	0.00627	0.00610	0.00580
ρ_{xTLh}	0.00738	0.00736	0.00553	0.00505	0.00661	0.00627	0.00610	0.00580
ρ_{yBLd}	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333
ρ_{yBLe}	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333	0.00333
ρ_{yTLb}	0.00607	0.00609	0.00596	0.00547	0.00333	0.00333	0.00333	0.00333
ρ_{yTLc}	0.00630	0.00629	0.00600	0.00548	0.00333	0.00333	0.00333	0.00333
A_{Sp2} (cm^2)	36.77	36.83	26.81	28.61	58.70	61.23	36.96	38.41
A_{sP3} (cm^2)	36.77	36.83	41.21	44.30	36.96	38.41	36.96	38.41
A_{sxBL} (cm^2)	47.80	47.85	55.79	58.20	43.77	44.85	45.68	46.80
A_{sxBLi} (cm^2)	47.80	47.85	55.79	58.20	43.77	44.85	45.68	46.80
A_{sxBLj} (cm^2)	47.80	47.85	55.79	58.20	43.77	44.85	45.68	46.80
A_{sxBT} (cm^2)	71.66	71.72	76.49	79.09	87.96	89.82	86.93	88.77
A_{sxTL} (cm^2)	105.80	105.67	92.61	88.23	86.82	84.37	83.56	81.41
A_{sxTLg} (cm^2)	103.62	103.50	92.58	88.21	86.82	84.37	83.56	81.41
A_{sxTLh} (cm^2)	105.80	105.67	92.61	88.23	86.82	84.37	83.56	81.41
A_{sxTT} (cm^2)	91.52	91.61	98.74	103.01	107.92	110.56	106.89	109.51
A_{syBL} (cm^2)	50.70	50.75	49.59	51.73	95.16	97.50	95.16	97.50
A_{syBLd} (cm^2)	50.70	50.75	49.59	51.73	95.16	97.50	95.16	97.50
A_{syBLe} (cm^2)	50.70	50.75	49.59	51.73	95.16	97.50	95.16	97.50
A_{syBT} (cm^2)	40.38	40.40	49.12	50.90	29.97	30.12	39.65	40.33
A_{syTL} (cm^2)	95.86	95.75	89.28	85.02	95.16	97.50	95.16	97.50
A_{syTLb} (cm^2)	92.25	92.67	88.63	84.83	95.16	97.50	95.16	97.50
A_{syTLc} (cm^2)	95.86	95.75	89.28	85.02	95.16	97.50	95.16	97.50
A_{syTT} (cm^2)	60.23	60.29	63.60	66.35	61.67	63.18	59.61	61.07
C_T	41.06C_c	41.07C_c	41.31C_c	42.09C_c	47.72C_c	48.64C_c	47.45C_c	48.38C_c

where: T = Theoretical, P = Practical

4 Results

Table 4 shows the results of the final design of the four examples (effective depth, total thickness, reinforcing steel areas, volume of concrete, volume of reinforcing steel, and total volume).

Table 4. Final design of the four examples of the corner combined footings

Concept	Example 1	Example 2	Example 3	Example 4
d (cm)	87.00	97.00	117.00	117.00
t (cm)	95.00	105.00	125.00	125.00
A_{Sp2} (cm^2)	40.56 (8Ø1”)	30.42 (6Ø1”)	65.91 (13Ø1”)	40.56 (8Ø1”)
A_{sP3} (cm^2)	40.56 (8Ø1”)	45.63 (9Ø1”)	40.56 (8Ø1”)	40.56 (8Ø1”)
A_{sxBL} (cm^2)	50.70 (10Ø1”)	60.84 (12Ø1”)	45.63 (9Ø1”)	50.70 (10Ø1”)
A_{sxBT} (cm^2)	74.10 (26Ø3/4”)	79.80 (28Ø3/4”)	91.20 (32Ø3/4”)	91.20 (32Ø3/4”)
A_{sxTL} (cm^2)	106.47 (21Ø1”)	91.26 (18Ø1”)	86.19 (17Ø1”)	86.19 (17Ø1”)
A_{sxTT} (cm^2)	94.05 (33Ø3/4”)	105.45 (37Ø3/4”)	111.15 (39Ø3/4”)	111.15 (39Ø3/4”)
A_{syBL} (cm^2)	55.77 (11Ø1”)	55.77 (11Ø1”)	101.40 (20Ø1”)	101.40 (20Ø1”)
A_{syBT} (cm^2)	42.75 (15Ø3/4”)	51.30 (18Ø3/4”)	31.35 (11Ø3/4”)	42.75 (15Ø3/4”)
A_{syTL} (cm^2)	96.33 (19Ø1”)	86.19 (17Ø1”)	101.40 (20Ø1”)	101.40 (20Ø1”)
A_{syTT} (cm^2)	62.70 (22Ø3/4”)	68.40 (24Ø3/4”)	65.55 (23Ø3/4”)	62.70 (22Ø3/4”)
V_c (m^3)	18.2433	19.8636	24.0308	24.0691
V_s (m^3)	0.2627	0.2554	0.2817	0.2809
V_t (m^3)	18.5060	20.1180	24.3125	24.3500
C_T	41.89C_c	42.85C_c	49.39C_c	49.35C_c

Table 2 shows the four examples to find the optimal area or minimum contact surface for the corner combined footings with the ground. The constant parameters for the four examples are: the axial loads (P_1, P_2 and P_3), the moments around the X axis (M_{x1}, M_{x2} and M_{x3}), the moment around the Y axis (M_{y1}, M_{y2} and M_{y3}), the sides of the columns (c_1, c_2, c_3 and c_4), the separation between columns (L_1 and L_2), and the available permissible load capacity of the soil (q_{aa}). The design variables to find are: the sides (a, b, b_1 and b_2), the moments of inertia around each axis (I_x and I_y), the resultant force (R), the resultant moments (M_{xT} and M_{yT}), the distance from the center of gravity to the furthest fiber in each direction (x_t, x_b, y_t and y_b), and the pressures at each vertex of the footing (q_1, q_2 q_3, q_4, q_5 and q_6), these variables are assumed non-negative (except for the moments). This table shows the following: 1) The smallest contact area is presented in examples 1, 2 and 3 of A_{min} = 18.48 m^2 (Theoretical), and in example 2 of A_{min} = 19.16 m^2 (Practical). 2) The pressure under the footing is uniform for examples 1, 2 and 3 (Theoretical), because the resultant moments M_{xT} and M_{yT} are zero, i.e., the resultant force of all the forces is located at the center of gravity of the footing. 3) The greatest contact area is presented in example 4 of A_{min} = 19.28 m^2 (Theoretical), and in examples 1 and 4 of A_{min} = 19.48 m^2 (Practical).

Table 3 shows the minimum cost for design, the effective depth, the percentages of reinforcing steel, and the reinforcing steel areas. The known parameters for the four examples are: the sides (a, b, b_1 and b_2), the factored moments (M_{ua}, M_{ub}, M_{uc}, M_{ud}, M_{ue}, M_{uf}, M_{ug}, M_{uh}, M_{ui} and M_{uj}), the factored bending shears (V_{uk}, V_{ul}, V_{um}, V_{un}, V_{uo}, V_{up}, V_{uq} and V_{ur}) are presented in function of “d”, the factored punching shears (V_{up1}, V_{up2} and V_{up3}) are presented in function of “d”, the maximum and minimum percentages. The design variables to find are: the effective depth (d), the percentages of reinforcing steel at each section (ρ_{P2}, ρ_{P3}, ρ_{xBLi}, ρ_{xBLj}, ρ_{xTLg}, ρ_{xTLh}, ρ_{yBLd}, ρ_{yBLe}, ρ_{yTLb} and ρ_{yTLc}), the reinforcing steel areas (A_{sP2}, A_{sP3}, A_{sxBL}, A_{sxBLi}, A_{sxBLj}, A_{sxBT}, A_{sxTL}, A_{sxTLg}, A_{sxTLh}, A_{sxTT}, A_{syBL}, A_{syBLd}, A_{syBLe}, A_{syBT}, A_{syTL}, A_{syTLb}, A_{syTLc} and A_{syTT}). This table shows the following: 1) The lowest cost is presented in example 1 of C_T = 41.06C_c (Theoretical), and also in example 1 of C_T = 41.07C_c (Practical). 2) The highest cost is presented in example 3 of C_T = 47.72C_c (Theoretical), and also in example 3 of C_T = 48.64C_c (Practical). 3) The smallest effective depth is presented in example 1 of d = 86.91 cm (Theoretical), and also in example 1 of d = 87.00 cm (Practical). 4) The greatest effective depth is presented in example 3 of d = 114.96 cm (Theoretical), and also in examples 3 and 4 of d = 117.00 cm (Practical).

Table 4 shows the final design. This table shows the following: 1) The smallest effective depth is presented in example 1 of d = 87.00 cm, and the greatest effective depth is presented in in examples 3 and 4 of d = 117.00 cm. 2) The smallest volume of concrete is presented in example 1 of V_c = 18.2433 m^3, and the greatest volume of concrete is presented in example 4 of V_c = 24.0691 m^3. 3) The smallest volume of steel is presented in example 2 of V_s = 0.2554 m^3, and the greatest volume of steel is presented in example 3 of V_s = 0.2817 m^3. 4) The smallest total volume is presented in example 1 of V_t = 18.5060 m^3, and the greatest total volume is presented in example 4 of V_t = 24.3500 m^3. 5) The lowest cost is presented in example 1 of C_T = 41.89C_c, and the highest cost is presented in example 3 of C_T = 49.39C_c.

Fig. 5 shows in detail the dimensions and the reinforcing steel for the corner combined footing in a general way.

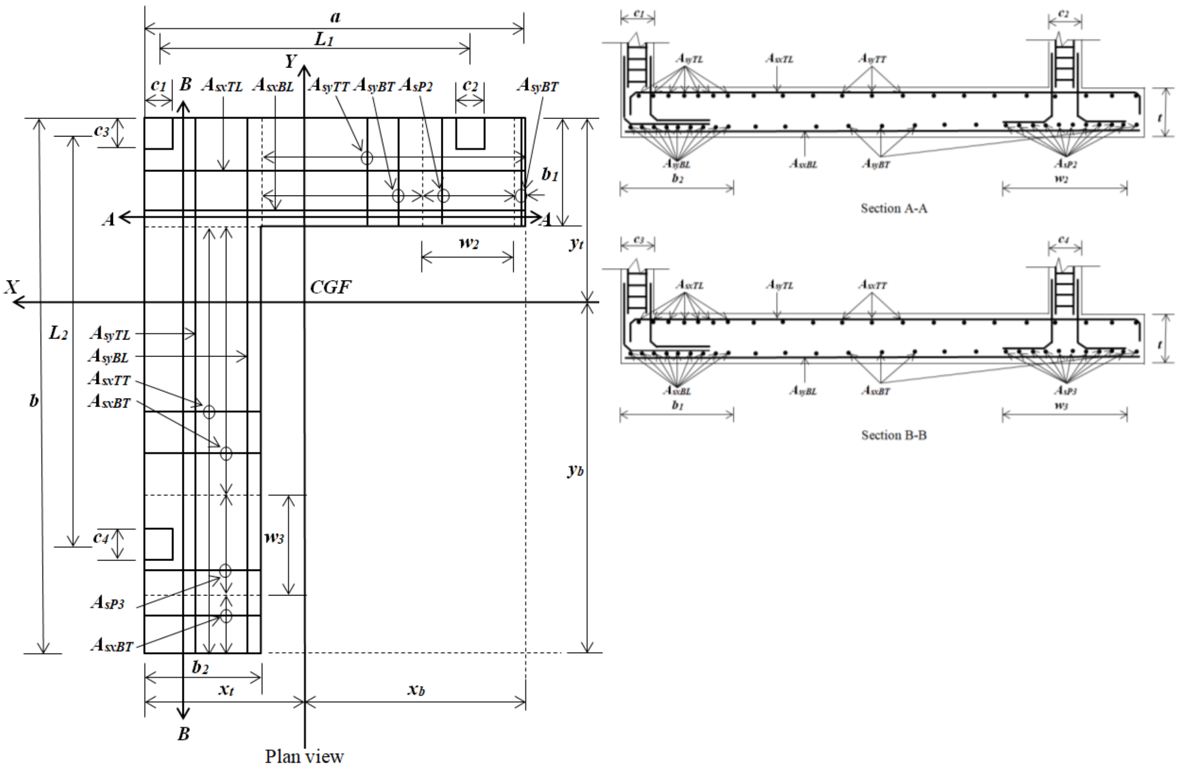

Fig. 5. General diagram for a corner combined footing

5 Conclusions

The model presented in this paper deals the design of minimum cost for reinforced concrete corner combined footings subjected to an axial load, a moment around the "X" axis and a moment around the "Y" axis, these effects are provided by each column. The optimal area or minimum contact surface considers the following: The constant or known parameters are: P_1, P_2, P_3, M_{x1}, M_{x2}, M_{x3}, M_{y1}, M_{y2}, M_{y3}, c_1, c_2, c_3, c_4, L_1, L_2, and q_{aa}. The decision or unknown variables are: A_{min}, a, b, b_1, b_2, I_x, I_y, R, M_{xT}, M_{yT}, x_t, x_b, y_t, y_b, q_1, q_2 q_3, q_4, q_5 and q_6.

The optimal model is shown to obtain the total minimum cost of the materials used (concrete and reinforcing steel) for the construction of the corner combined footings and the constraint functions are generated according to the requirements of the building code (ACI 318-19) [26].

The optimal design or minimum cost considers the following: The constant or known parameters are: a, b, b_1, b_2, c_1, c_2, c_3, c_4, H, L_1, L_2 , M_{ua}, M_{ub}, M_{uc}, M_{ud}, M_{ue}, M_{uf}, M_{ug}, M_{uh}, M_{ui}, M_{uj}, V_{uk}, V_{ul}, V_{um}, V_{un}, V_{uo}, V_{up}, V_{uq}, V_{ur}, V_{up1}, V_{up2}, V_{up3}, q_a, γ_c, γ_g, r, α, f'_c and f_y. The design or unknown variables are: C_T, d, ρ_{P2}, ρ_{P3}, ρ_{xBLi}, ρ_{xBLj}, ρ_{xTLg}, ρ_{xTLh}, ρ_{yBLd}, ρ_{yBLe}, ρ_{yTLb}, ρ_{yTLc}, A_{sP2}, A_{sP3}, A_{sxBL}, A_{sxBLi}, A_{sxBLj}, A_{sxBT}, A_{sxTL}, A_{sxTLg}, A_{sxTLh}, A_{sxTT}, A_{syBL}, A_{syBLd}, A_{syBLe}, A_{syBT}, A_{syTL}, A_{syTLb}, A_{syTLc} and A_{syTT}.

The proposed model presented in this paper concludes the following:

1. The optimal model is flexible and could be used for three or four property lines, because the values of "a" and/or "b" can be restricted on the corner combined footings. These values are not affected in the design, because simply M_{ue} and/or M_{uj} are equal to zero and V_{un} and/or V_{ur} does not exist.
2. The most economical design is presented in example 1 of C_T = 41.89C_c, because the dimensions "a" and "b" are not restricted (see Table 4).
3. The order of least to greatest of the examples investigated is:
    a) For the minimum contact area is 2, 3, 1 and 4 (Practical) (see Table 2).

   b) For the minimum cost for design is 1, 2, 4 and 3 (Practical) (see Table 4).
4. Therefore, there is no direct relationship between the optimal area and the minimum cost design.
5. The proposed methodology shown in this work is more economical, more precise and converges more quickly.
6. The objective function and constraint functions are shown by simplified and generalized equations.
7. The proposed model could be used for other concrete design codes, this can be done by changing the equations of the resistant moment, the resistant bending shear and the resistant punching shear according to the specifics of each code to obtain the minimum cost for the corner combined footings.

The proposed model presented in this paper for the structural design of corner combined footings subjected to an axial load and moment in two directions in each column can be applied to others cases: The footings subjected to a concentric axial load in each column, and the footings subjected to an axial load and moment in one direction in each column.

The suggestions for future research could be:
1) Optimal design of another type of structural foundation.
2) Optimal design of another type of structural members for reinforced concrete and structural steel.
3) Optimal design for the complete structure.

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